LESSON

9 The Product and Quotient Rules

▶ The Product Rule

When a function consists of parts that are added together, it is easy to take its derivative: Simply take the derivative of each part and add them together. We are inclined to try the same trick when the parts are multiplied together, but it does not work.

For example, we know that $\frac{d}{dx}(x^2) = 2x$ and $\frac{d}{dx}(x^3) = 3x^2$. The derivative of their product is $\frac{d}{dx}(x^2 \cdot x^3) = \frac{d}{dx}(x^5) = 5x^4$. This shows that the derivative of a product is *not* the product of the derivatives:

$$5x^4 = \frac{d}{dx}(x^2 \cdot x^3) \neq \frac{d}{dx}(x^2) \cdot \frac{d}{dx}(x^3) = (2x) \cdot (3x^2) = 6x^3$$

Instead, we take the derivative of each part, multiply by the other part *left alone*, and add the results together:

$$\frac{d}{dx}(x^2 \cdot x^3) = \frac{d}{dx}(x^2) \cdot x^3 + \frac{d}{dx}(x^3) \cdot x^2 = (2x) \cdot x^3 + (3x^2) \cdot x^2 = 5x^4$$

This time, we *did* get the correct answer.

The Product Rule

The Product Rule can be stated "the derivative of the first times the second, plus the derivative of the second times the first." It can be proven directly from the limit definition of the derivative, but only with a few tricks and a lot of algebra. The Product Rule is given as follows:

$$\frac{d}{dx}(f(x)\cdot g(x)) = f'(x)\cdot g(x) + g'(x)\cdot f(x)$$

Example

Differentiate $y = x^3\sin(x)$.

Solution

Here, the "first part" is x^3 and the "second part" is $\sin(x)$. Thus, by using the Product Rule, $\frac{d}{dx}(x^3\sin(x)) = \frac{d}{dx}(x^3)\cdot\sin(x) + \frac{d}{dx}(\sin(x))\cdot x^3 = 3x^2\sin(x) + \cos(x)\cdot x^3$. This could be simplified as $\frac{dy}{dx} = x^2(3\sin(x) + x\cos(x))$, but that's not really all that's necessary.

Example

Differentiate $f(x) = \ln(x)\cdot\cos(x)$.

Solution

$$f'(x) = \frac{d}{dx}(\ln(x))\cdot\cos(x) + \frac{d}{dx}(\cos(x))\cdot\ln(x)$$
$$= \frac{1}{x}\cdot\cos(x) - \sin(x)\cdot\ln(x)$$

Thus, the derivative is:

$$f'(x) = \frac{\cos(x)}{x} - \ln(x)\cdot\sin(x)$$

Example

Differentiate $g(x) = 5x^7\cdot e^x$.

Solution

$$g'(x) = \frac{d}{dx}(5x^7)\cdot e^x + \frac{d}{dx}(e^x)\cdot 5x^7$$
$$= 35x^6\cdot e^x + e^x\cdot 5x^7 = 5x^6e^x(7 + x)$$

Using the product rule with e^x can be a little bit confusing because there is no difference between the derivative of e^x and e^x "left alone." Still, if you write everything out, the correct answer should fall into place, even if it looks weird.

Example

Differentiate $y = t^2\ln(t)$.

Solution

$$\frac{dy}{dt} = \frac{d}{dt}(t^2)\ln(t) + \frac{d}{dt}(\ln(t))t^2$$
$$= 2t\cdot\ln(t) + \frac{1}{t}\cdot t^2$$
$$= 2t\cdot\ln(t) + t$$
$$= t(2 + \ln(t))$$

Example

Differentiate $y = x^5\sin(x)\cos(x)$.

Solution

We'll use the Product Rule with x^5 as the first part and $\sin(x)\cos(x)$ as the second part. However, in taking the derivative of $\sin(x)\cos(x)$, we'll have to use the

The Quotient Rule

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}$$

Product Rule a second time. It might get messy, but it will work if everything is written down carefully.

$$\frac{dy}{dx} = \frac{d}{dx}(x^5) \cdot \sin(x)\cos(x) + \frac{d}{dx}(\sin(x)\cos(x)) \cdot x^5$$

$$\frac{dy}{dx} = 4x^5\sin(x)\cos(x) + \left[\frac{d}{dx}(\sin(x)) \cdot \cos(x) + \frac{d}{dx}(\cos(x)) \cdot \sin(x)\right] \cdot x^5$$

$$\frac{dy}{dx} = 4x^5\sin(x)\cos(x) + \left[\cos(x) \cdot \cos(x) - \sin(x) \cdot \sin(x)\right] \cdot x^5$$

$$\frac{dy}{dx} = 4x^5\sin(x)\cos(x) + (\cos^2(x) - \sin^2(x)) \cdot x^5$$

▶ Practice

Differentiate the following.

1. $f(x) = x^2\cos(x)$
2. $y = 8t^3e^t$
3. $y = \sin(x)\cos(x)$
4. $g(x) = 3x^2\ln(x) - 5x^4 + 10$
5. $h(u) = ue^u - e^u$
6. $k(x) = \sin(x) + x^4 - x^2\sin(x)$
7. $y = 8\ln(x)\sin(x) + \cos(x)$
8. $h(t) = t\sin(t) - t\cos(t)$
9. $y = 5x^3 - x\ln(x)$
10. $f(x) = \sin^2(x) = \sin(x) \cdot \sin(x)$
11. $y = xe^x\sin(x)$
12. $g(x) = 3x^4\ln(x)\cos(x)$
13. What is the slope of the tangent line to $f(x) = x^2e^x + x + 2$ at $(0,2)$?
14. Find the equation of the tangent line to $y = x\sin(x)$ at $x = \pi$.

▶ The Quotient Rule

The Quotient Rule for functions where the parts are divided is even more complicated than the Product Rule. The Quotient Rule can be stated:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}$$

Just as with the Product Rule, each part is differentiated and multiplied by the other part. Here, however, they are subtracted, so it matters which one is differentiated first. It is important to start with the derivative of the top.

Example

Differentiate $y = \dfrac{x^5 - 3x^2 + 1}{\cos(x)}$.

Solution

Here, the top part is $x^5 - 3x^2 + 1$ and the bottom part is $\cos(x)$. Therefore, by the Quotient Rule:

$$\frac{dy}{dx} = \frac{\frac{d}{dx}(x^5 - 3x^2 + 1)\cdot\cos(x) - \frac{d}{dx}(\cos(x))\cdot(x^5 - 3x^2 + 1)}{(\cos(x))^2}$$

$$\frac{dy}{dx} = \frac{(5x^4 - 6x)\cdot\cos(x) - (-\sin(x))\cdot(x^5 - 3x^2 + 1)}{\cos^2(x)}$$

$$\frac{dy}{dx} = \frac{(5x^4 - 6x)\cdot\cos(x) + \sin(x)\cdot(x^5 - 3x^2 + 1)}{\cos^2(x)}$$

Example

Differentiate $f(x) = \dfrac{x^3}{10x^2 - 1}$.

Solution

$$f'(x) = \frac{\frac{d}{dx}(x^3)\cdot(10x^2 - 1) - \frac{d}{dx}(10x^2 - 1)\cdot x^3}{(10x^2 - 1)^2}$$

$$f'(x) = \frac{(3x^2)\cdot(10x^2 - 1) - (20x)\cdot x^3}{(10x^2 - 1)^2}$$

$$f'(x) = \frac{30x^4 - 3x^2 - 20x^4}{(10x^2 - 1)^2} = \frac{10x^4 - 3x^2}{(10x^2 - 1)^2}$$

Example

Differentiate $y = \dfrac{x^2\sin(x)}{\ln(x)}$.

Solution

Here, the Product Rule is necessary to differentiate the top.

$$\frac{dy}{dx} = \frac{\frac{d}{dx}(x^2\sin(x))\cdot\ln(x) - \frac{d}{dx}(\ln(x))\cdot x^2\sin(x)}{(\ln(x))^2}$$

$$\frac{dy}{dx} = \frac{\left[\frac{d}{dx}(x^2)\cdot\sin(x) + \frac{d}{dx}(\sin(x))\cdot x^2\right]\cdot\ln(x) - \frac{1}{x}\cdot x^2\sin(x)}{(\ln(x))^2}$$

$$\frac{dy}{dx} = \frac{\left[2x\cdot\sin(x) + \cos(x)\cdot x^2\right]\cdot\ln(x) - x\sin(x)}{(\ln(x))^2}$$

Example

Differentiate $y = \dfrac{\ln(t)}{t}$.

Solution

$$\frac{dy}{dt} = \frac{\frac{d}{dt}(\ln(t))\cdot t - \frac{d}{dt}(t)\cdot\ln(t)}{t^2}$$

$$\frac{dy}{dt} = \frac{\frac{1}{t}\cdot t - 1\cdot\ln(t)}{t^2}$$

$$\frac{dy}{dt} = \frac{1 - \ln(t)}{t^2}$$

Some people remember the Quotient Rule as

$$\frac{d}{dx}\left(\frac{HI}{HO}\right) = \frac{HO \cdot d(HI) - HI \cdot d(HO)}{HO \cdot HO}$$

just so they can say "HI d'HO" and "HO HO," but they're silly.

▶ Practice

Differentiate the following.

15. $h(x) = \dfrac{x^3 + 10x - 7}{3x^2 + 5x + 2}$

16. $y = \dfrac{4e^t + t}{t^3 + 2t + 1}$

17. $f(x) = \dfrac{x + \ln(x)}{e^x - 1}$

18. $y = \dfrac{x^5}{\ln(x)}$

19. $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$

20. $g(t) = \dfrac{t^3}{5\sin(t)}$

21. $y = \dfrac{x + 1}{x - 1}$

22. $g(u) = \dfrac{\sin(u)}{u^3 - e^u}$

23. $y = \dfrac{x^2 + 2x + e^x}{\sin(x) + 1}$

24. $h(t) = \dfrac{\ln(t) + t}{t^2}$

25. $y = \dfrac{x\ln(x)}{e^x}$

26. $f(x) = \dfrac{x^2 e^x}{\cos(x)}$

27. Find the second derivative of $y = \dfrac{x + 1}{x - 4}$.

28. What is the slope of the tangent line to $f(x) = \dfrac{x^2 + x}{x + 5}$ at $x = 5$?

▶ Derivatives of Trigonometric Functions

With the Quotient Rule, we can find the derivatives of all of the rest of the trigonometric functions.

Example

Differentiate $y = \tan(x)$.

Solution

Use $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$.

$$\frac{dy}{dx} = \frac{d}{dx}(\tan(x)) = \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right)$$

Differentiate with the Quotient Rule.

$$\frac{dy}{dx} = \frac{\cos(x) \cdot \cos(x) - (-\sin(x)) \cdot \sin(x)}{\cos^2(x)}$$

Simplify.

$$\frac{dy}{dx} = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}$$

Use $\sin^2(x) + \cos^2(x) = 1$.

$$\frac{dy}{dx} = \frac{1}{\cos^2(x)}$$

Use $\sec(x) = \dfrac{1}{\cos(x)}$.

$$\frac{dy}{dx} = \sec^2(x)$$

Thus:

$$\frac{d}{dx}(\tan(x)) = \sec^2(x)$$

Example

Differentiate $y = \sec(x)$.

Solution

Use $\sec(x) = \dfrac{1}{\cos(x)}$.

$$\frac{dy}{dx} = \frac{d}{dx}(\sec(x)) = \frac{d}{dx}\left(\frac{1}{\cos(x)}\right)$$

Differentiate with the Quotient Rule.

$$\frac{dy}{dx} = \frac{0 \cdot \cos(x) - (-\sin(x)) \cdot 1}{\cos^2(x)}$$

Simplify.

$$\frac{dy}{dx} = \frac{\sin(x)}{\cos^2(x)} = \frac{1}{\cos(x)} \cdot \frac{\sin(x)}{\cos(x)}$$

Use $\sec(x) = \dfrac{1}{\cos(x)}$ and $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$.

$$\frac{dy}{dx} = \sec(x)\tan(x)$$

Thus:

$$\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x)$$

▶ Practice

Differentiate the following.

29. $y = \csc(x)$

30. $y = \cot(x)$

31. $f(x) = x\tan(x)$

32. $g(x) = \dfrac{\sqrt{x}}{\sec(x)}$